International Journal of Theoretical Physics, Vol. 39, No. 6, 2000

Topos Perspective on the Kochen–Specker Theorem: III. Von Neumann Algebras as the Base Category

J. Hamilton,[1] C. J. Isham,[1] and J. Butterfield[2]

Received November 10, 1999

We extend the topos-theoretic treatment given in previous papers of assigning values to quantities in quantum theory, and of related issues such as the Kochen–Specker theorem. This extension has two main parts: the use of von Neumann algebras as a base category and the relation of our generalized valuations to (i) the assignment to quantities of intervals of real numbers and (ii) the idea of a subobject of the coarse-graining presheaf.

1. INTRODUCTION

Two previous papers [1, 2] have developed a topos-theoretic perspective on the assignment of values to quantities in quantum theory. In particular, it was shown that the Kochen–Specker theorem (which states the impossibility of assigning to each bounded self-adjoint operator on a Hilbert space of dimension greater than 2 a real number such that functional relations are preserved) is equivalent to the nonexistence of a global element of a certain presheaf $\mathbf{\Sigma}$, called the spectral presheaf, defined on the category $\mathcal{O}$ of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. In particular, the Kochen–Specker theorem's *FUNC* condition—which states that assigned values preserve the operators' functional relations—turns out to be equivalent to the 'matching condition' in the definition of a global section of the spectral presheaf. It was similarly shown that the Kochen–Specker theorem is equivalent to the

[1] The Blackett Laboratory, Imperial College of Science, Technology and Medicine, London SW7 2BZ, U.K; e-mail: j.hamilton@ic.ac.uk, c.isham@ic.ac.uk
[2] All Souls College, Oxford OX1 4AL, U.K.; e-mail: jb56@cus.cam.ac.uk; jeremy.butterfield@all-souls.oxford.ac.uk

0020-7748/00/0600-1413$18.00/0

nonexistence of a global element of a presheaf **D**—called the dual presheaf—defined on the category $\mathcal{W}$ of Boolean subalgebras of the lattice $\mathcal{L}(\mathcal{H})$ of projectors on $\mathcal{H}$.

It was also shown that it *was* possible to define so-called *generalized valuations* on all quantities according to which any proposition "$A \in \Delta$" (read as saying that the value of A lies in the Borel set of real numbers Δ) is assigned, in effect, a set of quantities that are coarse-grainings (functions) of A. To be precise, it is assigned a certain set of morphisms in the category $\mathcal{O}$ (or $\mathcal{W}$), the set being required to have the structure of a *sieve*. These generalized valuations obey a condition analogous to *FUNC* and other natural conditions. Furthermore, each (pure or mixed) quantum state defines such a valuation.

In this paper, we will extend this treatment in two main ways. The first corresponds to our previous concerns with the Kochen–Specker theorem and global sections and with generalized valuations based on sieves. Thus, we will first discuss the issues adumbrated above in terms of a base category different from $\mathcal{O}$ and $\mathcal{W}$: namely, the category $\mathcal{V}$ of commutative von Neumann subalgebras of an algebra of operators (Sections 2 and 3).

Second, we will further develop the idea of a generalized valuation (Section 4). In particular, we introduce the idea of an interval-valued valuation: at its simplest, the idea is to assign to a quantity A—not an individual member of its spectrum, as vetoed by the Kochen–Specker theorem—but rather, some subset of it. Though this idea seems at first sight very different from generalized valuations that assign sieves to propositions "$A \in \Delta$," we shall see that the two types of valuations turn out to be closely related.

2. VON NEUMANN ALGEBRAS

2.1. Introducing $\mathcal{V}$

We will first rehearse the definitions given in the previous papers [1, 2] of the categories $\mathcal{O}$ and $\mathcal{W}$ defined in terms of the operators on a Hilbert space over which various presheaves may be usefully constructed. Then we will introduce a new base category $\mathcal{V}$ which has as objects commutative von Neumann algebras and relate it to $\mathcal{O}$ and $\mathcal{W}$.

The categories $\mathcal{O}$ and $\mathcal{W}$ were defined as follows. The objects of the category $\mathcal{O}$ are the bounded self-adjoint operators on the Hilbert space $\mathcal{H}$ of some quantum system. A morphism $f_{\mathcal{O}}: \hat{B} \to \hat{A}$ is defined to exist if $\hat{B} = f(\hat{A})$ [in the sense of ref. 1, Eq. (2.4)] for some Borel function f. This category is a preorder, and may be turned into a partially ordered set by forming equivalence classes of operators. Operators $\hat{A}$ and $\hat{B}$ are considered equivalent whenever they are isomorphic in the category $\mathcal{O}$, i.e., when there exist some

Borel functions f and g such that $\hat{B} = f(\hat{A})$ and $\hat{A} = g(\hat{B})$. The category obtained in this way is denoted $[\mathcal{O}]$.

The category $\mathcal{W}$ is defined to have as its objects the Boolean subalgebras of the lattice $\mathcal{L}(\mathcal{H})$ of projectors on $\mathcal{H}$. A morphism is defined to exist from W_1 to W_2 if, and only if, $W_1 \subseteq W_2$. Thus $\mathcal{W}$ is just a partially ordered set (poset) equipped with the natural categorical structure of such a poset.

This category $\mathcal{W}$ is related to $\mathcal{O}$ via the covariant *spectral algebra functor* $\mathbf{W}: \mathcal{O} \to \mathcal{W}$, which is defined as follows:

- On objects: $\mathbf{W}(\hat{A}) := W_A$, where W_A is the spectral algebra of the operator $\hat{A}$ [i.e., the collection of all projectors onto the subspaces of $\mathcal{H}$ associated with Borel subsets of $\sigma(\hat{A})$].
- On morphisms: If $f_{\mathcal{O}}: \hat{B} \to \hat{A}$, then $\mathbf{W}(f_{\mathcal{O}}): W_B \to W_A$ is defined as the subset inclusion $i_{W_B W_A}: W_B \to W_A$.

Note that operators in the same equivalence class in $[\mathcal{O}]$ will always have the same spectral algebra.

We now wish to introduce a new base category $\mathcal{V}$ of commutative von Neumann algebras. We first recall (see, e.g., ref. 6) a few facts about von Neumann algebras.

A (not necessarily commutative) von Neumann algebra $\mathcal{N}$ is a C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$ which is closed in the weak operator topology. The algebra $\mathcal{N}$ is generated by its lattice of projectors $\mathcal{L}(\mathcal{N})$, and is equal to its own double commutant and to the double commutant of $\mathcal{L}(\mathcal{N})$:

$$\mathcal{L}(\mathcal{N})'' = \mathcal{N}'' = \mathcal{N} \tag{2.1}$$

The algebra contains all operators obtainable as Borel functions $f(\hat{A})$ of all normal operators $\hat{A} \in \mathcal{N}$.

We now define the category $\mathcal{V}$ associated with the Hilbert space $\mathcal{H}$ of some quantum system. The objects V in $\mathcal{V}$ are the commutative von Neumann subalgebras of the algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$, and the morphisms in $\mathcal{V}$ are the subset inclusions—so if $V_2 \subseteq V_1$, we have a morphism $i_{V_2 V_1}: V_2 \to V_1$. Thus the objects in the category $\mathcal{V}$ form a poset.

The category $\mathcal{V}$ is related to $\mathcal{O}$ via a covariant functor in a similar way to $\mathcal{W}$:

Definition 2.1. The *von Neumann algebra functor* is the covariant functor $\mathbf{V}: \mathcal{O} \to \mathcal{V}$ defined as follows:

- On objects: $\mathbf{V}(\hat{A}) := V[A]$, where $V[A]$ is the commutative von Neumann algebra generated by the self-adjoint operator $\hat{A}$.
- On morphisms: If $f_{\mathcal{O}}: \hat{B} \to \hat{A}$, then $\mathbf{V}(f_{\mathcal{O}}): V[B] \to V[A]$ is defined as the subset inclusion $i_{V_B V_A}: V[B] \to V[A]$.

There is an even simpler relation between $\mathcal{W}$ and $\mathcal{V}$:

Definition 2.2. The *algebra generation functor* is the covariant functor $\mathbf{V^W}: \mathcal{W} \to \mathcal{V}$ defined as follows:

- On objects: $\mathbf{V^W}(W) := W''$, where W'' is the double commutant of W in the algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, so that $\mathbf{V^W}(W)$ is the commutative von Neumann algebra generated by the projection operators in W.
- On morphisms: If $i_{W_2W_1}: W_2 \to W_1$, then $\mathbf{V^W}(i_{W_2W_1}): W_2'' \to W_1''$ is defined as the subset inclusion $i_{V_2V_1}: W_2'' \to W_1''$.

The category $\mathcal{V}$ seems to give the most satisfactory description of the ordering structure of operators. In particular, there is no problem with isomorphic operators: any operator isomorphic to $\hat{A}$ will always be included in any subalgebra which contains $\hat{A}$. Also, each von Neumann algebra contains the spectral projectors of all its self-adjoint members, so in a sense $\mathcal{V}$ subsumes both $\mathbb{O}$ and $\mathcal{W}$.

Using $\mathcal{V}$ as the base category is also appealing from an interpretative point of view. Many discussions and proofs of the Kochen–Specker theorem are written in terms of subalgebras of operators and their relations.[3]

2.2. Presheaves on $\mathcal{V}$

The spectral presheaf $\mathbf{\Sigma}$ over $\mathbb{O}$ was introduced in ref. 1. We now define the corresponding presheaf over $\mathcal{V}$ and the state presheaf over $\mathcal{V}$.

2.2.1. The Spectral Presheaf $\mathbf{\Sigma}$

We recall (see, e.g., ref. 6) that the spectrum $\sigma(V)$ of a commutative von Neumann algebra V is the set of all multiplicative linear functionals $\kappa: V \to \mathbb{C}$. Such a functional assigns a complex number $\kappa(\hat{A})$ to each operator $\hat{A} \in V$ such that $\kappa(\hat{A})\kappa(\hat{B}) = \kappa(\hat{A}\hat{B})$. If $\hat{A}$ is self-adjoint, $\kappa(\hat{A})$ is real and belongs to the spectrum of the operator $\hat{A}$ in the usual way.

Furthermore, $\sigma(V)$ is a compact Hausdorff space when it is equipped with the weak-* topology, which is the weakest topology such that, for all $\hat{A} \in V$, the map $\tilde{A}: \sigma(V) \to \mathbb{C}$ defined by

$$\tilde{A}(\kappa) := \kappa(\hat{A}) \tag{2.2}$$

is continuous. The quantity $\tilde{A}$ defined in Eq. (2.2) is known as the *Gelfand*

[3] In particular, Kochen and Specker in their original paper [3] formulate their theorem in terms of *partial algebras*, which have a similar category-theoretic structure. Some recent work on the 'modal interpretation' focuses on certain 'beable' subalgebras of operators as those on which valuations can be constructed [4, 5].

transform of $\hat{A}$, and the spectral theorem for commutative von Neumann algebras asserts that the map $\hat{A} \mapsto \tilde{A}$ is an isomorphism of V with the algebra $C(\sigma(V))$ of complex-valued, continuous functions on $\sigma(V)$.

Definition 2.3. The *spectral presheaf* over $\mathcal{V}$ is the contravariant functor $\mathbf{\Sigma}: \mathcal{V} \to \mathrm{Set}$ defined as follows:

- On objects: $\mathbf{\Sigma}(V) := \sigma(V)$, where $\sigma(V)$ is the spectrum of the commutative von Neumann algebra V, i.e., the set of all multiplicative linear functionals $\kappa: V \to \mathbb{C}$.
- On morphisms: If $i_{V_2 V_1}: V_2 \to V_1$, so that $V_2 \subseteq V_1$, then $\mathbf{\Sigma}(i_{V_2 V_1}): \sigma(V_1) \to \sigma(V_2)$ is defined by $\mathbf{\Sigma}(i_{V_2 V_1})(\kappa) := \kappa|_{V_2}$, i.e., this is the restriction of the functionals $\kappa: V_1 \to \mathbb{C}$ to V_2.

As discussed in ref. 1, for the base category $\mathcal{O}$, the Kochen–Specker theorem may be written in terms of the spectral presheaf. If it existed, a global element of $\mathbf{\Sigma}$ over $\mathcal{V}$ would assign a multiplicative linear functional $\kappa: V \to \mathbb{C}$ to each commutative von Neumann algebra V in $\mathcal{V}$ in such a way that these functionals match up as they are mapped down the presheaf. To be precise, the functional κ on V would be obtained as the restriction to V of the functional $\kappa_1: V_1 \to \mathbb{C}$ for any $V_1 \supseteq V$.

Furthermore, when restricted to the self-adjoint elements of V, a multiplicative linear functional κ satisfies all the conditions of a *valuation*, namely:

1. The (real) value $\kappa(\hat{A})$ of $\hat{A}$ must belong to the spectrum of $\hat{A}$.
2. The functional composition principle (*FUNC*)
$$\kappa(\hat{B}) = f(\kappa(\hat{A})) \tag{2.3}$$
holds for any self-adjoint operators $\hat{A}, \hat{B} \in V$ such that $\hat{B} = f(\hat{A})$.

The Kochen–Specker theorem, which states that no such valuations exist on all operators on a Hilbert space of dimension greater than two, can therefore be expressed as the statement that the presheaf $\mathbf{\Sigma}$ over $\mathcal{V}$ has no global elements. The matching condition outlined above therefore cannot be satisfied over the whole of $\mathcal{V}$.

It is worth noting that a spectral presheaf may be associated with any (noncommutative) von Neumann algebra $\mathcal{N}$ by first considering the poset $\mathcal{A}(\mathcal{N})$ of all of its commutative subalgebras as a category and then constructing the functor $\mathbf{\Sigma}: \mathcal{A} \to \mathrm{Set}$ in the above manner. Similar comments apply to the other constructions introduced in the rest of this section.

2.2.2. *The State Presheaf* S

A *state* ρ on a C^*-algebra $\mathcal{C}$ with unit 1 is a functional $\rho: \mathcal{C} \to \mathbb{C}$ that is

1. linear
2. positive, i.e., $\rho(AA^*) \geq 0$ for all $A \in \mathcal{C}$
3. normalized, so that $\rho(1) = 1$

(see, e.g., ref. 6, p. 255). The space S of all states on some C^*-algebra $\mathcal{C}$ of operators is a convex set whose extreme points are the *pure states*. If $\mathcal{C}$ is commutative, a state ρ is pure if and only if it is a multiplicative functional, i.e., $\rho(\hat{A}\hat{B}) = \rho(\hat{A})\rho(\hat{B})$ for all $\hat{A}, \hat{B} \in \mathcal{C}$. The set of pure states is therefore the spectrum of $\mathcal{C}$; in particular, the real number $\rho(\hat{A})$ will belong to the spectrum of $\hat{A}$ for all self-adjoint operators $\hat{A} \in \mathcal{C}$.

We now define the state presheaf.

Definition 2.4. The *state presheaf* **S** over $\mathcal{V}$ is the contravariant functor $\mathbf{S}: \mathcal{V} \to \mathrm{Set}$ defined as follows:

- On objects: $\mathbf{S}(V)$ is the space of states of the commutative von Neumann algebra V.
- On morphisms: If $i_{V_2V_1}: V_2 \to V_1$, so that $V_2 \subseteq V_1$, then $\mathbf{S}(i_{V_2V_1})$: $\mathbf{S}(V_1) \to \mathbf{S}(V_2)$ is the restriction of the state functionals in $\mathbf{S}(V_1)$ to V_2.

A global element of this presheaf is an assignment of a state to each commutative von Neumann subalgebra that is consistent in the sense that the state on any subalgebra V_1 may be obtained by restriction from any larger subalgebra, i.e., if $V_1 \subseteq V_2$ and $V_1 \subseteq V_3$, then the states ρ_2 on V_2 and ρ_3 on V_3 must agree on their common subalgebra, so that

$$\rho_2(\hat{A}) = \rho_3(\hat{A}) \tag{2.4}$$

for all $\hat{A} \in V_1$.

One way to achieve this is to take a state $\rho_{\mathcal{H}}$ on the (noncommutative) von Neumann algebra $B(\mathcal{H})$—for example, given by a density matrix—and assign to each commutative subalgebra V the state $\rho_{\mathcal{H}|V}$ obtained by the restriction of $\rho_{\mathcal{H}}$ to V. However, there may be other global elements of **S** which are not obtainable in this way.

The Kochen–Specker theorem tells us that the consistency condition above cannot be satisfied for an assignment of a *pure* state to each commutative subalgebra in $\mathcal{V}$, as this would also be a global element of $\mathbf{\Sigma}$, and hence would correspond to a global valuation.

3. SIEVE-VALUED GENERALIZED VALUATIONS

In Section 2, we described the prohibition on global assignments of real-number values to quantum-theoretic quantities (the Kochen–Specker theorem), in terms of the state and spectral presheaves on $\mathcal{V}$. In this section, we will describe some possible generalized valuations which are not excluded by the Kochen–Specker theorem.

These constructions will have certain properties which strongly suggest that they are appropriate generalizations of the idea of a valuation—in particular, they satisfy a functional composition principle analogous to Eq. (2.3).

Sieve-valued valuations with these properties were introduced in ref. 1 using the base categories $\mathcal{O}$ and $\mathcal{W}$. These were motivated by the observation that partial, real-number-valued valuations (i.e., defined on only some subset of quantities) give rise to such valuations, and it was shown that quantum states could be used to define such valuations. In ref. 2, these valuations were given more general motivations; in particular, classical physical analogues of the quantum-theoretic valuations were given.

This section extends the discussions in refs. 1 and 2. First, we adapt to $\mathcal{V}$ the definitions, results, and discussion of sieve-valued valuations. We will then look at other possible types of generalized valuation, namely those which are obtained by assigning to each quantity a *subset* of its spectrum. Some examples of these *interval*-valued valuations are exhibited and related to the previous sieve-valued valuations.

3.1. The Coarse-Graining Presheaf G

We will follow refs. 1 and 2 in assigning sieves primarily, not to quantities, but to propositions about the values of quantities. In refs. 1 and 2, sieves were assigned to propositions saying that the value of a quantity $\hat{A}$ lies in a Borel set $\Delta \subset \sigma(\hat{A})$, or, more precisely, to the mathematical representative of the proposition, a projector $\hat{E}[A \in \Delta]$ in the spectral algebra W_A of $\hat{A}$.

The coarse-graining presheaf was defined over $\mathcal{O}$ (ref. 1, Def. 4.3) to show the behavior of these propositions as they are mapped between the different stages of the presheaf. Thus the coarse-graining presheaf over $\mathcal{O}$ is the contravariant functor $\mathbf{G}: \mathcal{O} \to \mathrm{Set}$ defined as follows:

- On objects in $\mathcal{O}$: $\mathbf{G}(\hat{A}) := W_A$, where W_A is the spectral algebra of $\hat{A}$.
- 0n morphisms in $\mathcal{O}$: If $f_{\mathcal{O}}: \hat{B} \to \hat{A}$ [i.e., $\hat{B} = f(\hat{A})$], then $\mathbf{G}(f_{\mathcal{O}}): W_A \to W_B$ is defined as

$$\mathbf{G}(f_{\mathcal{O}})(\hat{E}[A \in \Delta]) := \hat{E}[f(A) \in f(\Delta)] \tag{3.1}$$

Note that the action of this presheaf coarsens propositions (and their associated projectors) since the function f will generally not be injective and so $\hat{E}[f(A) \in f(\Delta)] \geq \hat{E}[A \in \Delta]$.

There are some subtleties arising from the fact that for Δ a Borel subset of $\sigma(\hat{A})$, $f(\Delta)$ need not be Borel. These are resolved in ref. 1, Theorem (4.1), by using the fact that if $\hat{A}$ has a purely discrete spectrum [so that, in particular, $f(\Delta)$ *is* Borel], then

$$\hat{E}[f(A) \in f(\Delta)] = \inf\{\hat{Q} \in W_{f(A)} \subseteq W_A | \hat{E}[A \in \Delta] \leq \hat{Q}\} \tag{3.2}$$

where the infimum of projectors is taken in the (complete) lattice structure of all projectors on $\mathcal{H}$. For a general self-adjoint operator $\hat{A}$ we used this in ref. 1 to *define* the coarse-graining operation; in other words, the projection operator denoted by $\hat{E}[f(A) \in f(\Delta)]$ is *defined* using the right-hand side of Eq. (3.2).

This infimum construction is used again in ref. 1, Section 5.3, to define a corresponding coarse-graining presheaf over $\mathcal{W}$, and our construction of $\mathbf{G}$ over $\mathcal{V}$ is similar to this. Specifically, we define:

Definition 3.1. The *coarse-graining presheaf* over $\mathcal{V}$ is the contravariant functor $\mathbf{G}$: $\mathcal{V} \to$ Set defined as follows:

- On objects: $\mathbf{G}(V)$ is the lattice $\mathcal{L}(V)$ of projection operators in V.
- On morphisms: if $i_{V_2V_1}$: $V_2 \to V_1$, then $\mathbf{G}(i_{V_2V_1})$: $\mathcal{L}(V_1) \to \mathcal{L}(V_2)$ is the coarse-graining operation defined on $\hat{P} \in \mathcal{L}(V_1)$ by

$$\mathbf{G}(i_{V_2V_2})(\hat{P}) := \inf\{\hat{Q} \in \mathcal{L}(V_2) | \hat{P} \leq i_{V_2V_1}(\hat{Q})\} \tag{3.3}$$

where the infimum exists because $\mathcal{L}(V_2)$ is complete.

The coarse-graining presheaf will play a central role in the definition of various types of generalized valuation. To set the scene, we will now discuss how propositions about the quantum system behave when coarse-grained to different stages in the base category $\mathcal{V}$. As we will now see, this is more subtle than the analogous process using the base category $\mathbb{O}$ or $\mathcal{W}$ (discussed in refs. 1 and 2).

In $\mathbb{O}$, a proposition about a quantum system at some stage $\hat{A} \in \mathbb{O}$ is a statement "$A \in \Delta$" that the value of the quantity A lies in some Borel subset Δ of the spectrum $\sigma(\hat{A})$ of the self-adjoint operator $\hat{A}$ that represents A. This proposition is associated with the stage $\hat{A}$ in $\mathbb{O}$, and is represented by the projector $\hat{E}[A \in \Delta] \in W_A$, where W_A is the spectral algebra of $\hat{A}$. For any $f_{\mathbb{O}}$: $\hat{B} \to \hat{A}$, so that $\hat{B} = f(\hat{A})$, the coarse-graining operation acts on the projector as follows:

$$\mathbf{G}(f_{\mathbb{O}})(\hat{E}[A \in \Delta]) = \hat{E}[f(A) \in f(\Delta)] = \hat{E}[B \in f(\Delta)] \tag{3.4}$$

where $\hat{E}[B \in f(\Delta)]$ is understood in the sense explained above.

This idea of coarse-graining means that the proposition "$A \in \Delta$" at stage $\hat{A}$ in $\mathbb{O}$ 'changes' in two ways under coarse-graining. First, the associated projector may change—we can have $\hat{E}[f(A) \in f(\Delta)] > \hat{E}[A \in \Delta]$. The second change—which always occurs—is that the *stage* of the proposition changes: after coarse-graining, we have a proposition at stage $f(\hat{A})$ in $\mathbb{O}$, so that the representing projector $\hat{E}[f(A) \in f(\Delta)]$ is thought of as belonging to

$W_{f(A)}$ even if, in fact, $\hat{E}[f(A) \in f(\Delta)] = \hat{E}[A \in \Delta]$. In short: there is no difficulty in interpreting a proposition (or projector) at a stage as associated with that stage's quantity.

Similar remarks apply to $\mathcal{W}$. The main difference from $\mathcal{O}$ is that for $\mathcal{W}$, a stage corresponds to an equivalence class of quantities or operators (viz. under the relation of being functions of each other), not a single quantity or operator. This just means that we interpret a proposition "$A \in \Delta$", or projector $\hat{E}[A \in \Delta]$, as associated with the entire stage, i.e., the Boolean algebra W_A, not with the specific quantity A or the specific operator $\hat{A}$. But the main point remains the same: that with either $\mathcal{O}$ or $\mathcal{W}$, there is no difficulty in interpreting a proposition or projector at a stage as associated with that stage—be it a quantity/operator, or an equivalence class, or common spectral algebra, of such.

However, in $\mathcal{V}$, the role of propositions such as "$A \in \Delta$" and the corresponding projectors $\hat{E}[A \in \Delta]$ is not so clear. For any stage V with $\hat{A} \in V$, the projector $\hat{E}[A \in \Delta]$ will belong to the spectral algebra of many different operators in V. Indeed, it will generally also belong to the spectral algebra of many operators *not* in V, so we may write the projector as $\hat{E}[B \in \Delta_B]$ for a large number of operators $\hat{B}$ with corresponding sets $\Delta_B \subset \sigma(\hat{B})$ and with $\hat{B} \notin V$.

So intuition pulls in two directions. On one hand, similarly to $\mathcal{W}$ above, the fact that the objects of the base category $\mathcal{V}$ are not operators, but von Neumann algebras, prompts us to interpret a proposition "$A \in \Delta$" as associated with the entire stage, i.e., with the algebra V, rather than with the specific physical quantity A or the corresponding specific operator $\hat{A}$. On the other hand, similarly to $\mathcal{O}$ above, operators are *elements* of stages, so that when $\hat{A}$ does belong to V it seems natural to think of a proposition "$A \in \Delta$" at a stage V in terms of the operator $\hat{A}$.

We favor the former option, since the latter option faces difficulties when we consider coarse-graining to a stage $V_2 \subset V_1$ which may not contain the operator $\hat{A}$. To spell these out, let us start by noting that to understand how the proposition "$A \in \Delta$" at stage V_1 (or, more precisely, the projector $\hat{E}[\hat{A} \in \Delta] \in \mathcal{L}(V_1)$) coarse-grains to some stage $V_2 \subset V_1$ according to Definition 3.1,

$$\mathbf{G}(i_{V_2V_1})(\hat{E}[A \in \Delta]) = \inf\{\hat{Q} \in \mathcal{L}(V_2) | \hat{E}[A \in \Delta] \leq i_{V_2V_1}(\hat{Q})\} \quad (3.5)$$

we need in general to consider three possibilities according to whether or not $\hat{A}$ and the projector $\hat{E}[A \in \Delta]$ are in V_2:

1. $\hat{A} \in V_2$. In this case, $\hat{E}[A \in \Delta] \in \mathcal{L}(V_2) \subset V_2$, and so the projector coarse-grains to itself: $\mathbf{G}(i_{V_2V_1})(\hat{E}[A \in \Delta]) = \hat{E}[A \in \Delta]$. And since

$\hat{A} \in V_2$, it is natural to assign the same interpretation—as a proposition about the value of $\hat{A}$—to the coarse-grained projector.

2. $\hat{A} \notin V_2$, but $\hat{E}[A \in \Delta] \in V_2$. In this case, the projector $\hat{E}[A \in \Delta]$ still coarse-grains to itself, mathematically speaking. However, it is not clear how the projector at stage V_2 could be interpreted in terms of the value of $\hat{A}$, since $\hat{A}$ is not present at that stage.
3. $\hat{A} \notin V_2$ and $\hat{E}[A \in \Delta] \notin V_2$. In this case, the coarse-grained projector is not the same as the original one, and so the proposition associated with it will no doubt be different. It is again unclear how one could interpret the coarse-grained projector in terms of $\hat{A}$: indeed, there is in general no clear choice as to which operator in V_2 is to be the topic of the coarse-grained proposition.

In the light of these difficulties, we will instead adopt the first option above: we interpret a projector $\hat{P} \in \mathcal{L}(V_1)$ as a proposition about the entire stage V_1. Formally, we can make this precise in terms of the spectrum of the *algebra* V_1. That is to say, we note that:

- Any projector $\hat{P} \in \mathcal{L}(V_1)$ corresponds not only to a subset of the spectrum of individual operators $\hat{A} \in V$ (where $\hat{P} \in W_A$ so $\hat{P} = \hat{E}[A \in \Delta]$ for some $\Delta \subset \sigma(\hat{A})$), but also to a subset of the spectrum of the whole algebra V_1, namely, those multiplicative linear functionals κ: $V_1 \to \mathbb{C}$ such that $\kappa(\hat{P}) = 1$.
- Coarse-graining respects this interpretation in the sense that if we interpret $\hat{P} \in \mathcal{L}(V_1)$ as a proposition about the spectrum of the algebra V_1, then the coarse-graining of $\hat{P}$ to some $V_2 \subset V_1$ given by $\inf\{\hat{Q} \in \mathcal{L}(V_2) | \hat{P} \leq i_{V_2V_1}(\hat{Q})\}$ is a member of $\mathcal{L}(V_2)$, and so can be interpreted as a proposition about the spectrum of the algebra V_2.

This treatment of propositions as concerning the spectra of commutative von Neumann algebras, rather than individual operators, amounts to the semantic identification of all propositions in the algebra corresponding to the same mathematical projector. Thus, when we speak of a proposition "$A \in \Delta$" at some stage V, with $\hat{A} \in V$, we really mean the corresponding proposition about the spectrum of the whole algebra V defined using the projector $\hat{E}[A \in \Delta]$. In terms of operators, the proposition "$A \in \Delta$" is *augmented* by incorporating in it all corresponding propositions "$B \in \Delta_B$" about other operators $\hat{B} \in V$ such that the projector $\hat{E}[A \in \Delta]$ belongs to the spectral algebra of $\hat{B}$, and $\hat{E}[A \in \Delta] = \hat{E}[B \in \Delta_B]$.

Definition 3.2. The *augmented proposition* at stage V in $\mathcal{V}$ associated to the projector $\hat{P} \in V$ is the collection of all propositions of the form "$A \in \Delta$," where $\hat{A} \in V$ and the Borel set $\Delta \subset \mathbb{R}$ is such that $\hat{E}[A \in \Delta] = \hat{P}$.

These *augmented propositions* in $\mathcal{V}$ then coarse-grain in an analogous way to standard propositions in $\mathbb{O}$: the augmented proposition "$A \in \Delta$" at stage V_1 is coarse-grained to "$f(A) \in f(\Delta)$" and the result is an augmented proposition at the lower stage V_2.

In ref. 1, Section 4.2.3, it was noted that the coarse-graining presheaf over $\mathbb{O}$ was essentially the same as the presheaf $B\boldsymbol{\Sigma}$ over $\mathbb{O}$, which assigns to each $\hat{A}$ in $\mathbb{O}$ the Borel subsets of the spectrum of $\hat{A}$. The presheaf $B\boldsymbol{\Sigma}$ on $\mathbb{O}$ was essentially the Borel power object of $\boldsymbol{\Sigma}$ containing those subobjects of $\boldsymbol{\Sigma}$ which are formed of Borel sets of spectral values, with a projector $\hat{E}[A \in \Delta] \in \mathbf{G}(\hat{A})$ corresponding to the Borel subset $\Delta \subset \boldsymbol{\Sigma}(\hat{A})$. This connection between projectors and subsets of spectra carries over to the algebra case.

A projection operator $\hat{P} \in V$ corresponds to a subset of the spectrum of V, namely the set of multiplicative linear functionals κ on V such that $\kappa(\hat{P}) = 1$. Bearing in mind that, for each $\hat{A}$ in $\mathcal{B}(\mathcal{H})$, the function $\tilde{A}$: $\sigma(V) \mapsto \mathbb{C}$ given by the Gelfand transform of $\hat{A}$, $\tilde{A}(\kappa) := \kappa(\hat{A})$, is continuous, we see that the subset of the spectrum of $\sigma(V)$ that corresponds to $\hat{P}$ is *closed* in the compact Hausdorff topology of $\sigma(V)$.

In fact, we can say more than this since, by virtue of the spectral theorem for commutative C^*-algebras, the operator $\hat{P} \in V$ is represented by a function $\tilde{P}$: $\sigma(V) \to \mathbb{R}$. Since $\hat{P}^2 = \hat{P}$, we see that $\tilde{P}$ is necessarily the characteristic function, χ_P say, of some subset of V, namely the set of all multiplicative linear functionals κ on V such that $\kappa(\hat{P}) = 1$. However, by virtue of the spectral theorem, χ_P is in fact a *continuous* function from $\sigma(V)$ to $[0, 1] \subset \mathbb{R}$, and hence the subset concerned is both open and closed. Thus the subset of $\sigma(V)$ corresponding to a projection operator $\hat{P} \in V$ is a *clopen* subset in the spectral topology. Conversely, of course, each clopen subset of $\sigma(V)$ corresponds to a projection operator $\hat{P}$ whose representative function $\tilde{P}$ on $\sigma(V)$ is the characteristic function of the subset.

So in analogy with $B\boldsymbol{\Sigma}$ on $\mathbb{O}$, we may define a similar presheaf on $\mathcal{V}$, which we will denote $\mathrm{Clo}\boldsymbol{\Sigma}$:

- On objects: $\mathrm{Clo}\boldsymbol{\Sigma}(V)$ is defined to be the set of clopen subsets of the spectrum $\sigma(V)$ of the algebra V; each such clopen set is the set of multiplicative linear functionals κ such that $\kappa(\hat{P}) = 1$ for some projector $\hat{P} \in V$.
- On morphisms: for $V_2 \subset V_1$, we define

$$\mathrm{Clo}\boldsymbol{\Sigma}(i_{V_2V_1})(\{\kappa \in \sigma(V_1) | \kappa(\hat{P}) = 1\}) = \{\chi \in \sigma(V_2) | \chi(\mathbf{G}(i_{V_2V_1})(\hat{P})) = 1\} \tag{36}$$

There is an isomorphism between $\mathbf{G}$ and $\mathrm{Clo}\boldsymbol{\Sigma}$, and so we can think of $\mathbf{G}$ on $\mathcal{V}$ as being the 'clopen' power object of $\boldsymbol{\Sigma}$ on $\mathcal{V}$.

3.2. Sieve-Valued Generalized Valuations

In view of the discussion of propositions in the previous subsection, our definition of a sieve-valued generalized valuation for the base category $\mathcal{V}$ will define the valuations on projectors in the explicit context of an algebra V. That is to say, the truth-value associated with a projector $\hat{P}$ depends on the context of a particular algebra V containing $\hat{P}$. As in any topos of presheaves (ref. 1, Appendix), the subobject classifier $\mathbf{\Omega}$ in the topos $\mathrm{Set}^{\mathcal{V}^{\mathrm{op}}}$ is a presheaf of sieves. Since $\mathcal{V}$ is a poset, sieves may be identified with lower sets in the poset. We define $\mathbf{\Omega}$ as follows:

- On objects: $\mathbf{\Omega}(V)$ is the set of sieves in $\mathcal{V}$ on V. We recall that $\mathbf{\Omega}(V)$ has (i) a minimal element, the empty sieve, $0_V = \emptyset$, and (ii) a maximal element, the principal sieve, $\mathrm{true}_V = \downarrow_V := \{V' | V' \subseteq V\}$.
- On morphisms: $\mathbf{\Omega}(i_{V_2V_1})$: $\mathbf{\Omega}(V_1) \to \mathbf{\Omega}(V_2)$ is the pullback of the sieves in $\mathbf{\Omega}(V_1)$ along $i_{V_2V_1}$ defined by

$$\mathbf{\Omega}(i_{V_2V_1})(S) = i^*_{V_2V_1}(S) := \{i_{V_3V_2}: V_3 \to V_2 | i_{V_2V_1} \circ i_{V_3V_2} \in S\} \tag{3.7}$$

$$= \{V_3 \subset V_2 | V_3 \in S\} \tag{3.8}$$

for all sieves $S \in \mathbf{\Omega}(V_1)$.

Then we define:

Definition 3.3. A *sieve-valued generalized valuation* on the category $\mathcal{V}$ in a quantum theory is a collection of maps ν_V: $\mathcal{L}(V) \to \Omega(V)$, one for each 'stage of truth' V in the category $\mathcal{V}$, with the following properties:

(i) *Functional composition*:

For any $\hat{P} \in \mathcal{L}(\mathcal{V})$ and any $V' \subseteq V$ so that $i_{V'V}: V' \to V$

$$\nu_{V'}(\mathbf{G}(i_{V'V}(\hat{P})) = i^*_{V'V}(\nu_V(\hat{P})) \tag{3.9}$$

(ii) *Null proposition condition*:

$$\nu_V(\hat{0}) = 0_V \tag{3.10}$$

(iii) *Monotonicity*:

$$\text{If} \quad \hat{P}, \hat{Q} \in \mathcal{L}(V) \quad \text{with} \quad \hat{P} \leq \hat{Q}, \quad \text{then} \quad \nu_V(\hat{P}) \leq \nu_V(\hat{Q}) \tag{3.11}$$

We may wish to supplement this list with:

(iv) *Exclusivity*:

$$\text{If} \quad \hat{P}, \hat{Q} \in \mathscr{L}(V) \quad \text{with} \quad \hat{P}\hat{Q} = \hat{0}$$
$$\text{and} \quad \nu_V(\hat{P}) = \text{true}_V, \quad \text{then} \quad \nu_V(\hat{Q}) < \text{true}_V \tag{3.12}$$

(v) *Unit proposition condition*:

$$\nu_V(\hat{1}) = \text{true}_V \tag{3.13}$$

Note that in writing Eq. (3.9), we have employed Definition 3.1 to specify the coarse-graining operation in terms of an infimum of projectors, as motivated by Theorem 4.1 of ref. 1.

The topos interpretation of these generalized valuations remains as discussed in Section 4.2 of ref. 1 and Section 4 of ref. 2. Adapting the results and discussion to the category $\mathcal{V}$, we have in particular the result that because of the *FUNC* condition, Eq. (3.9), the maps N^{ν}_V: $\mathscr{L}(V) \to \Omega(V)$ defined at each stage V by

$$N^{\nu}_V(\hat{P}) = \nu_V(\hat{P}) \tag{3.14}$$

define a natural transformation N^{ν} from $\mathbf{G}$ to $\mathbf{\Omega}$. Since $\mathbf{\Omega}$ is the subobject-classifier of the topos of presheaves, $\text{Set}^{\mathcal{V}^{\text{op}}}$, these natural transformations are in one-to-one correspondence with subobjects of $\mathbf{G}$, so that each generalized valuation defines a subobject of $\mathbf{G}$. We will pursue this topic in more detail in Section 4.4.3.

3.3. Sieve-Valued Valuations Associated with Quantum States

We recall (for example, ref. 1, Definition 4.5) that each quantum state ρ defines a sieve-valued generalized valuation on $\mathcal{O}$ or $\mathcal{W}$ in a natural way. For example, on $\mathcal{O}$, the generalized valuation was defined as

$$\begin{aligned}\nu^{\rho}(A \in \Delta) &:= \{ f_{\mathcal{O}}: \hat{B} \to \hat{A} | \text{Prob}(B \in f(\Delta); \rho) = 1\} \\ &= \{ f_{\mathcal{O}}: \hat{B} \to \hat{A} \,| \text{tr}(\rho \hat{E}[B \in f(\Delta)]) = 1\}\end{aligned} \tag{3.15}$$

Thus the generalized valuation associates to the proposition all arrows in $\mathcal{O}$ along which the projector corresponding to the proposition coarse-grains to a projector which is ‘true’ in the usual sense of having a Born-rule probability equal to 1. This construction is easily seen to be a sieve, and satisfies conditions analogous to Eqs. (3.9)–(3.13) for a generalized valuation on $\mathcal{O}$ (ref. 1, Section 4.4).

We also recall that there is a one-parameter family of extensions of these valuations defined by relaxing the condition that the proposition coarse-grains along arrows in the sieve to a ‘totally true’ projector. That is to say, we can define the sieve

$$\nu^{\rho,r}(A \in \Delta) := \{ f_{\mathcal{O}}\colon \hat{B} \to \hat{A} \,|\, \text{Prob}(B \in f(\Delta); \rho) \geq r\}$$
$$= \{ f_{\mathcal{O}}\colon \hat{B} \to \hat{A} \,|\, \text{tr}(\rho\, \hat{E}[B \in f(\Delta)]) \geq r\} \tag{3.16}$$

where the proposition "$A \in \Delta$" is only required to coarse-grain to a projector that is true with some probability greater than r, where $0.5 \leq r \leq 1$.

Furthermore, if one drops the exclusivity condition, one can allow probabilities less than 0.5, i.e., $0 < r < 0.5$.

We now introduce the same kind of valuation using $\mathcal{V}$ as the base category.

As was discussed in Section 3.1 in relation to **G**, when using the base category $\mathcal{V}$ it is more natural to interpret a projector $\hat{P} \in \mathcal{L}(V)$ as a proposition about the spectrum of the commutative subalgebra V rather than about the value of just one operator. Such an *augmented proposition* at a stage V in $\mathcal{V}$ will correspond to a projector $\hat{P} \in \mathcal{L}(V)$ and can be thought of as the family of propositions "$A \in \Delta$" for all $\hat{A} \in V$ that have $\hat{P}$ as a member of their spectral algebra with $\hat{E}[A \in \Delta] = \hat{P}$.

So we define a sieve-valued generalized valuation associated with a quantum state ρ as follows:

Definition 3.4. The sieve-valued valuation $\nu^{\rho}_{V_1}$ of a projector $\hat{P} \in V_1$ associated with a quantum state ρ is defined by

$$\nu^{\rho}_{V_1}(\hat{P}) := \{i_{V_2V_1}\colon V_2 \to V_1 \,|\, \rho[\mathbf{G}(i_{V_2V_1})(\hat{P})] = 1\} \tag{3.17}$$

This assigns as the truth-value at stage V_1 of a projector $\hat{P} \in V_1$ a sieve on V_1 containing (morphisms to V_1 from) all stages V_2 at which $\hat{P}$ is coarse-grained to a projector which is 'totally true' in the usual sense of having Born-rule probability 1.

One readily verifies that Eq. (3.17) defines a generalized valuation in the sense of Definition 3.3. The verification is the same, *mutatis mutandis*, as for generalized valuations on $\mathcal{O}$ given in Section 4.4 of ref. 1. As an example, we take the functional composition condition. Again, this requires that the sieves pull back in the appropriate manner; if $V_2 \subseteq V_1$ and hence $i_{V_2V_1}\colon V_2 \to V_1$, then

$$i^*_{V_2V_1}(\nu^{\rho}_{V_1}(\hat{P})) := \{i_{V_3V_2}\colon V_3 \to V_2 \,|\, i_{V_2V_1} \circ i_{V_3V_2} \in \nu^{\rho}_{V_1}(\hat{P})\} \tag{3.18}$$
$$= \{i_{V_3V_2}\colon V_3 \to V_2 \,|\, \rho[\mathbf{G}(i_{V_2V_1} \circ i_{V_3V_2})(\hat{P})] = 1\} \tag{3.19}$$

whereas

$$\nu^{\rho}_{V_2}(\mathbf{G}(i_{V_2V_1})(\hat{P})) := \{i_{V_3V_2}\colon V_3 \to V_2 \,|\, \rho[\mathbf{G}(i_{V_3V_2})(\mathbf{G}(i_{V_2V_1})(\hat{P}))] = 1\} \tag{3.20}$$

and hence *FUNC* is satisfied since $\mathbf{G}(i_{V_2V_1} \circ i_{V_3V_2})(\hat{P}) = \mathbf{G}(i_{V_3V_2})(\mathbf{G}(i_{V_2V_1})(\hat{P}))$.

Again, we can obtain a one-parameter family of such valuations by introducing a probability r:

$$\nu^{\rho,r}_{V_1}(\hat{P} := \{i_{V_2V_1}: V_2 \to V_1 | \rho\, [\mathbf{G}(i_{V_2V_1})\,(\hat{P})] \geq r\} \tag{3.21}$$

4. INTERVAL-VALUED GENERALIZED VALUATIONS

4.1. Introducing Interval-Valued Valuations

The sieve-valued generalized valuations on $\mathcal{V}$ discussed in Section 3 and their analogues on $\mathbb{O}$ and $\mathcal{W}$ (discussed in refs. 1 and 2) are one way of assigning a generalized truth value to propositions in a way that is not prevented by the Kochen–Specker theorem. We are now going to investigate another possibility—of assigning sets of real numbers to operators—and relate it to our generalized valuations. An algebra V in $\mathcal{V}$ will be assigned a subset of its spectrum, i.e., a set of multiplicative linear functionals on V which corresponds to a subset of the spectrum of each operator in the algebra.

We will call such assignments 'interval valuations' as the primary motivation for these assignments is the wish to assign some interval of real numbers to each operator. Note that, in the latter context, 'interval' means just some subset of $\mathbb{R}$, that is, an interval in our sense need not be a connected subset.

Despite the marked difference between interval-valued and sieve-valued valuations—projectors or propositions versus algebras as arguments, and sieves versus sets of real functionals as values—it turns out that these two kinds of valuations are closely related.

4.2. Subobjects of $\boldsymbol{\Sigma}$

We now consider assigning to each algebra V in $\mathcal{V}$ a clopen subset $I(V) \subseteq \sigma(V)$. This set $I(V)$ of multiplicative linear functionals κ on V leads to an assignment to each self-adjoint operator $\hat{A} \in V$ of a subset of the spectrum of the operator, $\Delta_A = \{\kappa(\hat{A}) | \kappa \in I(V)\} \subseteq \sigma(\hat{A})$, in such a way that the appropriate relationships between these subsets are obeyed, so that if $\hat{B} = f(\hat{A})$, we have $\Delta_B = f(\Delta_A)$. For we note that Δ_A is equal to $\tilde{A}[I(V)]$, where the Gelfand transform $\tilde{A}: \sigma(V) \to \mathbb{R}$ is defined in Eq. (2.2). However, $I(V)$ is a closed subset of the compact Hausdorff space $\sigma(V)$, and hence is itself compact. Then, since $\tilde{A}: \sigma(V) \to \mathbb{R}$ is continuous, it follows that $\tilde{A}[I(V)]$ is also compact; thus Δ_A is a *compact* subset of $\mathbb{R}$.

On the face of it, the fact that Δ_A is compact might appear problematic for functions $f: \sigma(\hat{A}) \to \mathbb{R}$ that are Borel but not continuous (since the image of a compact set by a Borel function need not itself be compact). In effect, the problem is that $f \circ \tilde{A}: \sigma(V) \to \mathbb{R}$ may not be continuous, even though it is supposed to represent the operator $\hat{B} = f(\hat{A})$. However, a more careful study of the spectral theorem for a commutative von Neumann algebra shows that the Borel function $f \circ \tilde{A}$ can be replaced by a unique continuous function (i.e., $\tilde{B}$) without changing anything of significance in the algebraic structure.[4] In addition, there exists some continuous function $\tilde{f}: \mathbb{R} \to \mathbb{R}$ such that $\tilde{B} = \tilde{f} \circ \tilde{A}$, and the natural-looking equation $\Delta_B = f(\Delta_A)$ is to be understood as $\Delta_B = \tilde{f}(\Delta_A)$ where necessary.

One of course expects to require such an interval-valued assignment I to obey some version of *FUNC* along morphisms in the category $\mathcal{V}$. The most obvious version is, for $V_2 \subset V_1$,

$$I(V_2) = I(V_1)|_{V_2} \tag{4.1}$$

so that the functionals in $I(V_2)$ are just the restriction to V_2 of those in $I(V_1)$.

But from the perspective of the theory of presheaves, it is natural to take such assignments I as given by subobjects of $\mathbf{\Sigma}$. These clearly exist—for example, there is the trivial subobject assigning to each algebra the whole of its spectrum. One can think of the Kochen–Specker theorem as restricting the 'smallness' of the subobjects: we cannot take a subobject that consists of a singleton set at each stage, since that would be a global element. If $\mathbf{I}$ is a subobject of $\mathbf{\Sigma}$, it will obey (by the definition of 'subobject') a *weaker* version of *FUNC*, viz.,

$$\mathbf{I}(i_{V_2 V_1})\,(\mathbf{I}(V_1)) := \mathbf{I}(V_1)\,|_{V_2} \subseteq \mathbf{I}(V_2) \tag{4.2}$$

Example 1. The 'true' Subobject of $\mathbf{\Sigma}$ *Arising from a Quantum State.* One subset of the spectrum of an operator $\hat{A}$ which arises naturally from a quantum state ρ is the set of values which can occur in a measurement of $\hat{A}$ when the system is in the state ρ. This set Δ_A is given in the following way.

For any pair of projectors $\hat{E}[A \in \Delta], \hat{E}[A \in \Delta'] \in W_A$ such that $\mathrm{tr}(\rho\hat{E}[A \in \Delta]) = \mathrm{tr}(\rho\hat{E}[A \in \Delta']) = 1$, there is a smaller projector $\hat{E}[A \in \Delta''] = \hat{E}[A \in \Delta]\hat{E}[A \in \Delta']$ (where $\Delta'' = \Delta \cap \Delta'$) such that $\mathrm{tr}(\rho\hat{E}[A \in \Delta'']) = 1$, and so we may form a descending net of projectors whose infimum exists and belongs to W_A, since W_A is complete. Thus there is a smallest projector $\hat{E}_{\min}$ which is of the form $\hat{E}[A \in \Delta_A]$ for some subset Δ_A of the spectrum of $\hat{A}$. Furthermore, the net of projectors converges strongly (and therefore weakly) to $\hat{E}[A \in \Delta_A]$, and therefore, since, for fixed ρ, the map $B(\mathcal{H}) \to \mathbb{C}$ given by $\hat{A} \to \mathrm{tr}(\rho\hat{A})$ is weakly continuous, it follows that $\mathrm{tr}(\rho\hat{E}[A \in \Delta_A]) = 1$. This operator $\hat{E}_{\min} = \hat{E}[A \in \Delta_A]$ is sometimes called the *support* of $\hat{A}$ in the state ρ.

For an algebra V, we can define the corresponding subset of its spectrum by taking those multiplicative linear functionals on V which assign the value

[4] For example, see the discussion on p. 324 in ref. 6.

1 to each 'true projector', i.e., those projectors in $T^\rho(V) := \{\hat{P} \in V | \mathrm{tr}(\rho\hat{P}) = 1\} \subset \mathcal{L}(V)$. The connection to the above assignment to operators is clear; for if $\hat{A} \in V$, then $\Delta_A = \{\kappa(\hat{A}) | \kappa(\hat{P}) = 1, \forall \hat{P} \in T^\rho(V)\}$.

So the corresponding 'true subobject' of $\mathbf{\Sigma}$ is given by

$$\mathbf{I}^\rho(V) = \{\kappa \in \sigma(V) | \kappa(\hat{P}) = 1, \forall \hat{P} \in T^\rho(V)\} \tag{4.3}$$

Again, since the map $\hat{P} \rightarrow \mathrm{tr}(\rho\hat{P})$ is weakly continuous, the infimum $\hat{Q} := \inf\{\hat{P} \in T^\rho(V)\}$ is in $T^\rho(V)$. Since $\kappa(\hat{Q}) = 1$ implies $\kappa(\hat{P}) = 1$ for all $\hat{P} \in T^\rho(V)$, the above construction may be written as the clopen set

$$\mathbf{I}^\rho(V) := \{\kappa \in \sigma(V) | \kappa(\hat{Q}) = 1\} \tag{4.4}$$

and so for nontrivial $\hat{Q}$ this will necessarily be a proper subset of $\sigma(V)$ (cf. the discussion at the end of Section 3.1 on the connection between projectors and subsets of spectra, in relation to Clo$\mathbf{\Sigma}$).

The above construction describes a subobject of $\mathbf{\Sigma}$ since, if we have $V_2 \subset V$, then

$$\mathbf{I}^\rho(V_2) = \{\kappa \in \sigma(V_2) | \kappa(\hat{P}) = 1, \forall\, \hat{P} \in T^\rho(V_2)\} \tag{4.5}$$

and since $T^\rho(V_2) = \{\hat{P} \in V_2 | \mathrm{tr}(\rho\hat{P}) = 1\} \subseteq T^\rho(V)$, we have that $\hat{Q}_2 \geq \hat{Q}$. Therefore $\{\kappa | \kappa(\hat{Q}) = 1\} \subseteq \{\kappa | \kappa(\hat{Q}_2) = 1\}$, and hence $\mathbf{I}^\rho(V)|_{V_2} \subset \mathbf{I}^\rho(V_2)$ as required for the subobject condition (4.2) to hold.

Example 2. Subobjects of $\mathbf{\Sigma}$ *from Sieve-Valued Valuations.* Given an extra condition on a sieve-valued valuation ν, the above construction of the 'true subobject' of $\mathbf{\Sigma}$ can be adapted to use ν to define an interval-valuation I^ν via

$$\mathbf{I}^\nu(V) = \{\kappa \in \sigma(V) | \kappa(\hat{P}) = 1, \forall\, \hat{P} \in T^\nu(V)\} \tag{4.6}$$

where now we define

$$T^\nu(V) := \{\hat{P} \in \mathcal{L}(V) | \nu_V(\hat{P}) = \mathrm{true}_V\} \tag{4.7}$$

We now look at the conditions under which this gives a subobject of $\mathbf{\Sigma}$. We note first that Eq. (4.6) is not in general a proper subset of $\Sigma(V)$. It will be empty unless the infimum $\hat{Q}$ of the set of projectors $T^\nu(V)$ is nonzero; for we can again write $\mathbf{I}^\nu(V)$ in the form of the clopen set

$$\mathbf{I}^\nu(V) = \{\kappa \in \sigma(V) | \kappa(\hat{Q}) = 1\} \tag{4.8}$$

which must be empty if $\hat{Q} = \hat{0}$, the zero projector.

The condition on ν for this construction to be a subobject is a matching condition on this infimum:

$$\text{For} \quad V_1 \subset V \quad \text{we require that} \quad \hat{Q}_1 \geq \hat{Q} \tag{4.9}$$

where $\hat{Q}$ and $\hat{Q}_1$ are the infima of the sets $T^\nu(V)$ and $T^\nu(V_1)$, respectively, defined as above. For, if this is not the case, there will be some $\kappa \in \sigma(V)$ such that $\kappa(\hat{Q}) = 1$ [and hence $\kappa \in \mathbf{I}^{\nu}(V)$] but $\kappa(\hat{Q}_1) < 1$. Therefore $\kappa|_{V_1} \notin \mathbf{I}^{\rho}(V_1)$, and the subobject condition (4.2) will not be satisfied.

A sufficient condition for Eq. (4.9) to hold is that if $V_1 \subset V$, then $T^{\nu}(V_1) \subseteq T^{\nu}(V)$. This condition is satisfied for the valuation ν^{ρ} arising from a quantum state in the probability one case via Eq. (3.15), and the resulting construction is, of course, the same as Example 1.

The condition also holds for valuations from a quantum state $\nu^{\rho,r}$ where a probability r is introduced, as in Eq. (3.21). In that case, however, there will be many algebras $V \in \mathcal{V}$ where there is no nontrivial infimum to the set $T^{\nu}(V)$, and so the corresponding subobject of $\mathbf{\Sigma}$ will be the empty set over much of $\mathcal{V}$.

4.3. Global Elements of G

As was discussed at the end of Section 3.1, there is a natural interpretation of the coarse-graining presheaf $\mathbf{G}$ as a subobject of the power object of $\mathbf{\Sigma}$, i.e., a presheaf of subobjects of $\mathbf{\Sigma}$.

Given this interpretation of $\mathbf{G}$, it is natural to take interval-valued valuations as given by global elements of $\mathbf{G}$. A global element γ of $\mathbf{G}$ obeys, for $V_2 \subset V_1$,

$$\gamma(V_2) = \mathbf{G}(i_{V_2V_1})\,(\gamma(V_1)) \tag{4.10}$$

which is a *FUNC* condition with an equality like Eq. (4.1) [as opposed to the subset version (4.2) required for subobjects of $\mathbf{\Sigma}$].

Recalling that for any projector selected by γ at stage V_1, i.e., $\gamma(V_1) = \hat{P} \in \mathcal{L}(V_1)$, the action of $\mathbf{G}$ is $\mathbf{G}(i_{V_2V_1})$: $\hat{P} \mapsto \inf\{\hat{Q} \in \mathcal{L}(V_2) | \hat{P} \leq i_{V_2V_1}(\hat{Q})\}$, we see that any such global element γ defines an assignment I^{γ} to each algebra of a subset of its spectrum by

$$I^{\gamma}\,(V_1) := \{\kappa \in \sigma(V_1) | \kappa(\gamma(V_1)) = 1\} \tag{4.11}$$

and, since $\hat{P} \leq \mathbf{G}(i_{V_2V_1})(\hat{P})$, we see that $\kappa(\hat{P}) = 1$ implies $\kappa(\mathbf{G}(i_{V_2V_1})(\hat{P})) = 1$, so that $I^{\gamma}(V_1)|_{V_2} \subseteq I^{\gamma}(V_2)$, and hence this construction is a subobject of $\mathbf{\Sigma}$ as well as an interval-valued valuation satisfying the stronger version of *FUNC*, Eq. (4.1).

In this way, an interval-valued valuation—assigning a subset of the spectrum at each stage—is given by every global element of $\mathbf{G}$.

As in Example 2 of the previous subsection, certain sieve-valued valuations can be used to give global elements of $\mathbf{G}$. The natural way to define such a global element is by

$$\gamma^\nu(V_1) := \inf\{\hat{P} \in \mathcal{L}(V_1) | \nu_{V_1}(\hat{P}) = \text{true}_{V_1}\} \tag{4.12}$$

$$=: \hat{Q}_1 = \inf\{\hat{P} | \hat{P} \in T^\nu(V_1)\} \tag{4.13}$$

where again, as in Eq. (4.7), $T^\nu(V_1) := \{\hat{P} \in \mathcal{L}(V_1) | \nu_{V_1}(\hat{P}) = \text{true}_{V_1}\}$.

The matching condition for this to be a global element of **G** is stronger than for it to be a subobject of **Σ**: we now require that for every $V_2 \subset V_1$,

$$\inf\{\hat{P} \in T^\nu(V_2)\} =: \hat{Q}_2 = \mathbf{G}(i_{V_2 V_1})(\inf\{\hat{P} \in T^\nu(V_1)\}) = \mathbf{G}(i_{V_2 V_1})(\hat{Q}_1) \tag{4.14}$$

The case of a valuation arising from a quantum state for the probability one case—Example 1 in the previous subsection—may be cast in this form. For that case we have that if $V_2 \subset V_1$, then $T^\rho(V_2) = \{\hat{P} \in V_2 | \text{tr}(\rho\hat{P}) = 1\} \subseteq T^\rho(V_1)$, and so each $\hat{P}_2 \in V_2$ is given by $\mathbf{G}(i_{V_2V_1})(\hat{P}_1)$ for some $\hat{P}_1 \in V_1$, since in particular this is true when $\hat{P}_1$, $\hat{P}_2$ denote the same projector thought of as belonging to V_1 and V_2, respectively. Then, since $\hat{P}_1 \leq \hat{P}'_1$ implies $\mathbf{G}(i_{V_2V_1})(\hat{P}_1) \leq \mathbf{G}(i_{V_2V_1})(\hat{P}'_1)$, it follows that $\inf\{\hat{P} \in T^\rho(V_2)\} = \mathbf{G}(i_{V_2V_1})(\inf\{\hat{P} \in T^\rho(V_1)\})$, and so we have a global element of **G**.

The valuations $\nu^{\rho,r}$ arising from a quantum state via the probability r construction [Eq. (3.21)], while giving subobjects of **Σ**, do not satisfy the stronger condition necessary to form global elements of **G**.

One way to avoid these issues arising from having to take infima of certain sets of projectors is to look at subobjects of **G**, rather than its global elements.

4.4. Subobjects of G

As described at the end of Section 3.2, every sieve-valued valuation ν on projectors induces a morphism N^ν: $\mathbf{G} \to \mathbf{\Omega}$ in the topos of presheaves over $\mathcal{V}$, and hence corresponds to a subobject of **G**. This subobject $\mathbf{T}^\nu$ is given at each stage V by the set

$$\mathbf{T}^\nu(V) = \{\hat{P} \in \mathcal{L}(V) | \nu_V(\hat{P}) = \text{true}_V\} \tag{4.15}$$

This is the same construction as Eq. (4.7), but now considered as defining a subobject of **G**. In this subsection we discuss such subobjects, in particular those arising from sieve-valued valuations associated with a quantum state ρ for the probability 1 and probability r cases, as defined in Eqs. (3.17) and (3.21).

4.4.1. The Probability 1 Case

To obtain a subobject of **G** from a quantum state ρ, we take at each stage V_1 the subset of the lattice of projectors $\mathcal{L}(V_1)$,

$$\mathbf{T}^{\rho}(V_1) := \{\hat{P} \in V_1 | \mathrm{tr}(\rho\hat{P}) = 1\} \subset \mathscr{L}(V_1) \tag{4.16}$$

This is, of course, the subobject arising via Eq. (4.15) from the sieve-valued valuation ν^{ρ} associated with the quantum state ρ, defined in Eq. (3.17).

This forms a subobject of the coarse-graining presheaf since, if $i_{V_2V_1}: V_2 \to V_1$, then for any $\hat{P} \in \mathscr{L}(V_1)$

$$\mathrm{tr}(\rho(\mathbf{G}(i_{V_2V_1})(\hat{P}))) \geq \mathrm{tr}(\rho\hat{P}) \tag{4.17}$$

Hence it follows that

$$\mathbf{G}(i_{V_2V_1})(\mathbf{T}^{\rho}(V_1)) \subseteq \mathbf{T}^{\rho}(V_2) \tag{4.18}$$

and therefore $\mathbf{T}^{\rho}$ is a genuine subobject of $\mathbf{G}$. Also, $\mathscr{L}(V_2) \subseteq \mathscr{L}(V_1)$ and for all $\hat{P}_2 \in \mathscr{L}(V_2)$, $\mathbf{G}(i_{V_2V_1})(\hat{P}_1) = \hat{P}_2$, where $\hat{P}_1$, $\hat{P}_2$ denote the same projector $\hat{P}$ thought of as belonging to $\mathscr{L}(V_1)$ and $\mathscr{L}(V_2)$, respectively. It follows that for all $\hat{P}_2 \in \mathbf{T}^{\rho}(V_2)$ we have $\hat{P}_1 \in \mathbf{T}^{\rho}(V_1)$, and therefore

$$\mathbf{T}^{\rho}(V_2) \subseteq \mathbf{G}(i_{V_2V_1})(\mathbf{T}^{\rho}(V_1)) \tag{4.19}$$

and so this subobject obeys a functional composition principle $\mathbf{T}^{\rho}(V_2) = \mathbf{G}(i_{V_2V_1})(\mathbf{T}^{\rho}(V_1))$ with an equality.

In this particular case, we can use the subobject $\mathbf{T}$ of $\mathbf{G}$ to give at each stage a subset $I^{\rho}(V)$ of the spectrum of the von Neumann algebra V via

$$I^{\rho}(V) := \{\kappa \in \mathbf{\Sigma}(V) | \kappa(\hat{P}) = 1, \forall\hat{P} \in \mathbf{T}^{\rho}(V)\} \subset \mathbf{\Sigma}(V) \tag{4.20}$$

Since the lattice $\mathscr{L}(V)$ is complete, there exists a smallest projector in the set $\mathbf{T}^{\rho}(V)$ as in the case of the global element of $\mathbf{G}$: $\hat{Q} := \inf\{\hat{P} | \hat{P} \in \mathbf{T}^{\rho}(V)\}$. This is then the single projector at each stage defined by Eq. (4.13) for the case where ν is the sieve-valuation associated with the state ρ, and so the global element construction may be recovered in this case.

This then corresponds to a single augmented proposition at each stage, namely the collection of propositions "$A \in \Delta$" for each operator $\hat{A} \in V$ and Borel subset Δ such that $\hat{E}[A \in \Delta] = \hat{Q}$.

4.4.2. *The Probability* r *Case*

As discussed in Section 4.3, not all sieve-valuations lead to global elements of $\mathbf{G}$ or to subobjects of $\mathbf{\Sigma}$. In particular, there is no global element of $\mathbf{G}$ associated with the probability r case for a quantum state ρ. However, we can take a subobject of $\mathbf{G}$:

$$\mathbf{T}^{\rho,r}(V) := \{\hat{P} \in V | \mathrm{tr}(\rho\hat{P}) \geq r\} \tag{4.21}$$

where $0 \leq r \leq 1$. Heuristically, this is the set of propositions in $\mathscr{L}(V)$ which are assigned a probability greater than r. Equation (4.17) still holds, and hence this still gives a genuine subobject of $\mathbf{G}$.

The propositions $\{\hat{P} \in V | \mathrm{tr}(\rho\hat{P}) \geq r\}$ do not in general have a nontrivial infimum, (except when $r = 1$), and the infima will not obey the necessary matching conditions to allow us to construct a global element of $\mathbf{G}$ in the same way as was possible in the $r = 1$ case.

4.4.3. Semantic Subobjects

This discussion of certain subobjects of $\mathbf{G}$ induced via Eq. (4.15) by generalized valuations ν prompts us to ask what characterizes subobjects of $\mathbf{G}$ which can be induced in this way. The defining conditions for a generalized valuation ν, Eqs. (3.9)–(3.12), give four properties of the corresponding subobject $\mathbf{T}^{\nu}$ of $\mathbf{G}$ defined according to Eq. (4.15):

1. *Functional composition*: This ensures that we have a genuine subobject $\mathbf{T}^{\nu}$ of $\mathbf{G}$, since it implies in particular that for all $\hat{P} \in \mathbf{T}^{\nu}(V_1)$, so that $\nu(\hat{P}) = \mathrm{true}_{V_1}$, it is true that $\nu_{V_2}(\mathbf{G}(i_{V_2V_1})(\hat{P})) = \mathrm{true}_{V_2}$, so $\mathbf{G}(i_{V_2V_1})(\mathbf{T}^{\nu}(V_1)) \subseteq \mathbf{T}^{\nu}(V_2)$.
2. *Null proposition condition*: This implies that the always-false projector $\hat{0}_V$ never belongs to $\mathbf{T}^{\nu}(V)$.
3. *Monotonicity*: For $\hat{P}, \hat{Q} \in \mathscr{L}(V)$ with $\hat{P} \leq \hat{Q}$, this condition implies that if $\hat{P} \in \mathbf{T}^{\nu}(V)$, then also $\hat{Q} \in \mathbf{T}^{\nu}(V)$; so $\mathbf{T}^{\nu}(V)$ is required to be an upper set in $\mathscr{L}(V)$.
4. *Exclusivity*: If $\hat{P}, \hat{Q} \in \mathscr{L}(V)$ with $\hat{P}\hat{Q} = \hat{0}$ and $\hat{P} \in \mathbf{T}^{\nu}(V)$, then $\hat{Q} \notin \mathbf{T}^{\nu}(V)$.

Note that properties 1 and 2 are consequences of Eqs. (3.9) and (3.10) respectively, and weaker than them. On the other hand, properties 3 and 4 are equivalent to Eqs. (3.11) and (3.12), respectively.

We may wish to relax the exclusivity condition here (and the corresponding condition for generalized valuations) depending on the type of valuation being studied. As has already been pointed out, the exclusivity condition is not satisfied (and would not be expected to hold) for valuations where the probability required for a proposition to belong to the corresponding subobject of $\mathbf{G}$ is less than a half.

Given properties 1–4, we can then turn this around, and define a *semantic subobject* of $\mathbf{G}$ as being a subobject satisfying these properties. Semantic subobjects then form a set of possible generalized valuations for our quantum theory.

4.5. Interval-Valued Valuations from Ideals

We now present another way of obtaining an interval-valued valuation on $\mathcal{V}$ which does not rely on the use of sieve-valuations and the coarse-graining presheaf.

We recall that the closed ideals in a commutative von Neumann algebra V are in one-to-one correspondence with the closed subsets of the spectrum $\sigma(V)$ of V. More precisely, according to the spectral theorem, the algebra V is isomorphic to the algebra $C(\sigma(V))$ of continuous, complex-valued functions on its spectrum; and a closed ideal in the algebra corresponds to the set of all functions in $C(\sigma(V))$ that vanish on the associated closed subset of $\sigma(V)$.

Furthermore, according to a general result about extremely disconnected Hausdorff spaces [which $\sigma(V)$ is], each closed subset of $\sigma(V)$ differs from a unique clopen set by a meagre set. Therefore, any assignment of a closed ideal to each algebra in $\mathcal{V}$ will produce an interval-valued valuation assigning to each V in $\mathcal{V}$ a clopen subset Ξ of its spectrum. The functions in the ideal are then precisely those that are disjoint to the characteristic function of Ξ, i.e., those operators in V orthogonal to the projector $\hat{P}$ that corresponds to the characteristic function of Ξ.

Furthermore, there is a natural way to make such assignments of closed ideals to each algebra V in $\mathcal{V}$. We note that an ideal ι in any noncommutative von Neumann algebra $\mathcal{N}$ will induce an ideal $\iota_V \subset V$ in each of its commutative subalgebras V by restriction: $\iota_V = \iota \cap V$. In this way, the ideal ι assigns to each V the clopen subset of $\sigma(V)$ associated with ι_V.

We note that the lattice of closed, two-sided ideals in any noncommutative algebra $\mathcal{N}$ is an example of a quantale [7]. This structure can be viewed as the analogue of a spectrum for the noncommutative von Neumann algebra $\mathcal{N}$, and as we have just shown, it represents another natural collection of interval-valued valuations in our framework.

Example. The subset ι^ψ of the set $B(\mathcal{H})$ of bounded operators on some Hilbert space $\mathcal{H}$ defined by

$$\iota^\psi = \{\hat{A} \in B(\mathcal{H}) | \hat{A}\psi = 0\} \tag{4.22}$$

forms a closed left ideal since $\hat{B}\hat{A}\psi = 0$ for all $\hat{B} \in B(\mathcal{H})$, $\hat{A} \in \iota^\psi$. This induces a two-sided ideal in each commutative von Neumann subalgebra V:

$$\iota^\psi(V) = \{\hat{A} \in V | \hat{A}\psi = 0\} \tag{4.23}$$

We will now show that this ideal corresponds to the 'true' subobject of $\boldsymbol{\Sigma}$ described in Example 1 of Section 4.2 for the quantum state ψ.

We will denote the set of projectors in $\iota^\psi(V)$ by $\mathcal{P}(\iota^\psi(V))$. If the projectors $\hat{Q}_1, \hat{Q}_2 \in \mathcal{P}(\iota^\psi(V))$, so that $\hat{Q}_1\psi = 0 = \hat{Q}_2\psi$, then we also have $(\hat{Q}_1 + \hat{Q}_2)\psi = 0$, so $(\hat{Q}_1 + \hat{Q}_2) \in \mathcal{P}(\iota^\psi(V))$. It follows that there exists a largest projector $\hat{Q}_{\max} \in \iota^\psi(V)$

$$\hat{Q}_{\max} = \sup \mathcal{P}(\iota^\psi(V)) \tag{4.24}$$

As was noted above, a projector in V corresponds to a characteristic function in $C(\sigma(V))$ of some clopen subset $\Xi \subset \sigma(V)$, and the set of functions disjoint to a characteristic function are the functions in the ideal. The ideal in V is then associated with the subset of $\sigma(V)$ for which the projector $(\hat{1} - \hat{Q}_{\max})$ corresponds to the characteristic function, i.e., the set $\{\kappa \in \sigma(V) | \kappa(\hat{1} - \hat{Q}_{\max}) = 1\}$. We also note that $(\hat{1} - \hat{Q}_{\max})\psi = \psi$, and so $\langle\psi| (\hat{1} - \hat{Q}_{\max})|\psi\rangle = 1$, and indeed $(\hat{1} - \hat{Q}_{\max})$ is the smallest projector with this property. This therefore corresponds to the subobject of $\mathbf{\Sigma}$ given by $\mathbf{I}^{\rho}$ in Eq. (4.4) for the state ρ on V given by $\rho(\hat{A}) = \langle\psi|\hat{A}|\psi\rangle$.

5. CONCLUSIONS

In this paper, we have extended the topos-theoretic perspective on the assignment of values to quantities in quantum theory to the base category $\mathcal{V}$ of commutative von Neumann algebras, a category which generalizes the categories of contexts $\mathcal{O}$ and $\mathcal{W}$ used in refs. 1 and 2. As we have seen, the main results of refs. 1 and 2, both the leading ideas and the mathematical constructions, can be adapted to the von Neumann algebra case. (Though we have not spelt out all the details of this adaptation piece by piece, one can check the details for the topics we have omitted—for example, classical analogues—and general motivations for sieve-valued valuations, as discussed in ref. 2).

This adaptation of our results to $\mathcal{V}$ is straightforward, except that, as noted in Section 3.1, we need to be careful about two issues: (i) about interpreting a proposition "$A \in \Delta$" relative to a von Neumann algebra V as a context, and (ii) about the properties of spectral topologies. The upshot is that (i) we interpret a projector $\hat{P}$ at a context V in terms of all "$A \in \Delta$" with $A \in V$ and $E[A \in \Delta] = \hat{P}$, and (ii) $\mathbf{G}$ is isomorphic to the clopen power object Clo$\mathbf{\Sigma}$ of the spectral presheaf $\mathbf{\Sigma}$.

Accordingly, we conclude that the base category $\mathcal{V}$ of commutative von Neumann algebras is as natural a basis for developing the topos-theoretic treatment of the values of quantities in quantum theory, as were our previous categories $\mathcal{O}$ and $\mathcal{W}$; indeed, in certain respects $\mathcal{V}$ is a better basis, since it includes the others in a natural way. In particular, it is a natural basis for addressing the various topics listed in Section 6 of ref. 1.

In this paper, we have addressed one such topic, using $\mathcal{V}$: that of interval-valued valuations. As we saw in Section 4, there are close connections between our sieve-valued valuations and the assignment to quantities of subsets of their (operators') spectra, and so also the assigments to commutative algebras of subsets of their spectra. We displayed these connections in various ways: in terms of subobjects of $\mathbf{\Sigma}$, in terms of global elements of $\mathbf{G}$, and in terms of subobjects of $\mathbf{G}$. One main idea in making these connections was the set $T^{\nu}(V)$ of projectors $\hat{P} \in V$ that are "wholly true" according to the sieve-valued valuation ν, which for the case of ν given by a quantum state ρ is essentially the familiar quantum-theoretic idea of the support of a state. Finally, we noted in Section 4.5 that once we use $\mathcal{V}$ rather than $\mathcal{O}$ or $\mathcal{W}$ as our base category of contexts, we can use ideals in the von Neumann algebras to define interval-valued valuations in a natural way, though again, care is needed about the spectral topologies.

REFERENCES

1. C. J. Isham and J. Butterfield (1998). A topos perspective on the Kochen–Specker theorem: I. Quantum states as generalised valuations. *Int. J. Theor. Phys.* **37**, 2669–2733.
2. J. Butterfield and C. J. Isham (1999). A topos perspective on the Kochen–Specker theorem: II. Conceptual aspects, and classical analogues. *Int. J. Theor. Phys.* **38**, 827–859.
3. S. Kochen and E. P. Specker (1967). The problem of hidden variables in quantum mechanics. *J. Math. Mech.* **17**, 59–87.
4. R. Clifton. Beables in algebraic quantum theory. In *From Physics to Philosophy*, J. Butterfield and C. Pagonis, eds. (Cambridge University Press, Cambridge 1999), pp. 12–44.
5. H. Halvorson and R. Clifton (1999). Maximal beable subalgebras of quantum-mechanical observables, *Int. J. Theor. Phys.* **38**, 2441–2484.
6. R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras Volume 1: Elementary Theory* (Academic Press, New York, 1983).
7. C. Mulvey and J. W. Pelletier (1999). On the quantisation of points. *J. Pure Appl. Algebra*.